The constraint satisfaction problem for bounded width and Maltsev algebras

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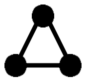
Constraint satisfaction problem (CSP)

Definition. For a finite relational structure $\mathbb{B} = (B; \mathcal{R})$ we define

$$\mathrm{CSP}(\mathbb{B}) = \{\, \mathbb{A} \mid \mathbb{A} \to \mathbb{B} \,\}.$$

Example. CSP() *is the class of three-colorable (directed) graphs.*

Example. CSP() *is the class of (directed) bipartite graphs.*

The membership problem for $\mathrm{CSP}(\mathbb{B})$ is always decidable in nondeterministic polynomial time (**NP**, intractable), sometimes in polynomial time (**P**, tractable).

Dichotomy Conjecture (Feder, Vardi, 1999)**.** *For every finite structure* $\mathbb{B}$ *the membership problem for* $\mathrm{CSP}(\mathbb{B})$ *is either in* **P** *or* **NP***-complete.*

Has been verified in many special cases (2-element structures, undirected graphs, smooth directed graphs, etc.) and yielded structure theorems in the tractable cases. Open for directed graphs.

CSP REDUCTIONS

Lemma. *We may assume, that*

- $\mathbb{B}$ *is a* **core**, *i.e., every endomorphism is an automorphism,*
- *every unary constraint relation* $\{b\}$ *is in* $\mathbb{B}$,
- *all relations are at most binary (or directed graph).*

Definition. A **polymorphism** of $\mathbb{B}$ is a homomorphism $p : \mathbb{B}^n \to \mathbb{B}$ (edge preserving operation).

$$\mathrm{Pol}(\mathbb{B}) = \{\, p \mid p : \mathbb{B}^n \to \mathbb{B} \,\}.$$

Lemma. $\mathrm{Pol}(\mathbb{B})$ *is a clone, and all polymorphisms are idempotent under our assumptions*

$$p(x, \ldots, x) \approx x.$$

Lemma. $\mathrm{Pol}(\mathbb{C}) \subseteq \mathrm{Pol}(\mathbb{B}) \implies \mathrm{CSP}(\mathbb{B})$ *is polynomial time reducible to* $\mathrm{CSP}(\mathbb{C})$.

- $\mathbb{B}$ has nice polymorphisms $\implies \mathrm{CSP}(\mathbb{B})$ is in **P**.
- $\mathbb{B}$ has no nice polymorphisms $\implies \mathrm{CSP}(\mathbb{B})$ is **NP**-complete.

Nice polymorphisms

Theorem. CSP($\mathbb{B}$) *is in* **P** *if* Pol($\mathbb{B}$) *contains one of the following:*

- *a semilattice operation* (Jevons et. al.)
- *a near-unanimity operation*

$$p(y, x, \ldots, x) \approx p(x, y, x, \ldots, x) \approx \cdots \approx p(x, \ldots, x, y) \approx x,$$

- *a totally symmetric idempotent operation* (Dalmau, Pearson, 1999),
- *a Maltsev operation:* $p(x, y, y) \approx p(y, y, x) \approx x$ (Bulatov, 2002; Dalmau, 2004),
- *Generalized majority-minority operation* (Dalmau, 2005),
- *Edge operations* (Idziak, Marković, McKenzie, Valeriote, Willard, 2007),
- *CD Jónsson operations* (Barto, Kozik, 2008),
- *SD*($\wedge$) *Willard operations* (Barto, Kozik, 2009),

Weak near-unanimity

Theorem (McKenzie, Maróti, 2006). *For a locally finite variety $\mathcal{V}$ the followings are equivalent:*

(1) *$\mathcal{V}$ omits type* **1**,

(2) *$\mathcal{V}$ has a Taylor term,*

(3) *$\mathcal{V}$ has a* **weak near-unanimity** *operation:*

$$p(y,x,\dots,x) \approx \dots \approx p(x,\dots,x,y) \quad \textit{and} \quad p(x,\dots,x) \approx x.$$

Theorem (Larose, Zádori, 2006). *If $\mathbb{B}$ is a core and does not have a Taylor (or weak near-unanimity) polymorphism, then* $\mathrm{CSP}(\mathbb{B})$ *is* **NP**-*complete.*

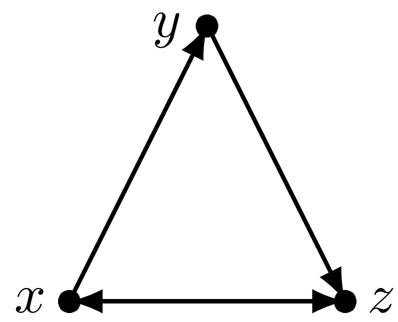
Dichotomy Conjecture. *If $\mathbb{B}$ is a core and has a weak near-unanimity polymorphism, then* $\mathrm{CSP}(\mathbb{B})$ *is in* **P**.

Applications of CSP to universal algebra

Theorem (Siggers, 2008)**.** *A locally finite variety $\mathcal{V}$ omits type* **1** *iff it has a 4-ary term t satisfying the equations*

$$t(x,y,z,x) \approx t(y,z,x,z) \quad \textit{and} \quad t(x,x,x,x) \approx x.$$

Proof. Consider the directed graph $\mathbb{G}$ defined on the 3-generated free algebra $\mathbf{F}_3(\mathcal{V})$ whose edges are generated by (x,y), (y,z), (z,x), (x,z). It is smooth, and its core must be a loop. That loop edge is $t((x,y),(y,z),(z,x),(x,z))$.



□

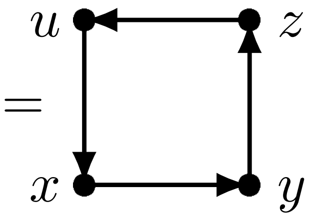
Theorem (Barto, Kozik, 2009)**.** *A locally finite variety $\mathcal{V}$ omits type* **1** *iff it has a cyclic term p satisfying the equations*

$$p(x_1,x_2,\dots,x_n) \approx \cdots \approx p(x_2,\dots,x_n,x_1) \quad \textit{and} \quad p(x,\dots,x) \approx x.$$

Theorem (Barto, 2009)**.** *A finite relational structure has a near-unanimity polymorphism if and only if it has Jónsson polymorphisms.*

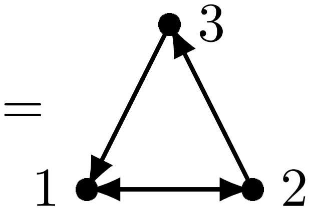
Consistency algorithm

$\mathbb{A} =$



$$\mathbb{A} \stackrel{?}{\in} \mathrm{CSP}(\mathbb{B})$$

$\mathbb{B} =$



$$\exists x, y, z, u \in \{1, 2, 3\} \quad (x, y) \in \mathbb{B} \wedge (y, z) \in \mathbb{B} \wedge (z, u) \in \mathbb{B} \wedge (u, x) \in \mathbb{B}$$

Strategies

Definition. A is a set, $\mathbb{B} = (B; \mathcal{R})$ is relational structure, $\mathcal{R}$ has at most binary relations and is closed under primitive positive formulas. A collection

$$\mathcal{B} = \{ B_{ij} \in \mathcal{R} \mid i, j \in A \}$$

of relations is a

- **strategy** if $B_{ji} = B_{ij}^{-1}$ and $B_{ii} \subseteq \{ (b, b) \mid b \in B \}$,
- **(1,2)-strategy** if $\pi_1(B_{ij}) = \pi_1(B_{ii})$ and $\pi_2(B_{ij}) = \pi_2(B_{jj})$,
- **(2,3)-strategy** if $B_{ik} \subseteq B_{ij} \circ B_{jk}$.

Definition. A function $f : A \to B$ is a **solution** of the strategy $\mathcal{B}$ if $(f(i), f(j)) \in B_{ij}$ for all $i, j \in A$.

Definition. The **local consistency algorithm** turns a strategy (or an instance of the CSP) into a (2,3)-strategy without loosing solutions:

$$B'_{ik} = B_{ik} \cap (B_{ij} \circ B_{jk}).$$

Bounded width

Lemma. *The local consistency algorithm*

- *runs in polynomial time (in the size of* $\mathbb{A}$*),*
- *the output is independent of the choices made,*
- *if the output strategy is empty, then* $\mathbb{A} \notin \mathrm{CSP}(\mathbb{B})$*.*

Definition. $\mathbb{B}$ has **width** $(2,3)$ if every nonempty $(2,3)$-strategy has a solution. The notion of **bounded width** is slightly more general.

Lemma. *If* $\mathbb{B}$ *has bounded width, then* $\mathrm{CSP}(\mathbb{B})$ *is in* **P***, but not conversely.*

Theorem (Larose, Zádori, 2006)**.** *If* $\mathbb{B}$ *has bounded width, then the variety generated by the algebra* $\mathbf{B} = (B; \mathrm{Pol}(\mathbb{B}))$ *omits types* **1** *and* **2***, i.e.,* $\mathbb{B}$ *has Willard polymorphisms.*

Theorem (Barto, Kozik, 2009)**.** $\mathbb{B}$ *has bounded width if and only if the variety generated by the algebra* $\mathbf{B} = (B; \mathrm{Pol}(\mathbb{B}))$ *omits types* **1** *and* **2***.*

If $\mathbb{B}$ has at most binary relations, then the $(2,3)$ local consistency algorithm works.

Maltsev algorithm

Definition. $\mathbf{B}$ is Maltsev if it has a term t satisfying the equations

$$t(x, y, y) \approx t(y, y, x) \approx x.$$

Definition. Let $\mathbf{P} \leq \mathbf{B}^n$.

- **index** is $(i, a, b) \in \{1, \ldots, n\} \times B \times B$,
- **witness** is $(\bar{a}, \bar{b}) \in P^2$ such that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$ and $a_i = a$ and $b_i = b$.
- **compact representation** is a collection of witnesses for each index that can be witnessed.

Given an element $\bar{d} \in \mathbf{B}^n$ and an approximation $\bar{c} \in \mathbf{P}$:

$$c_1 = d_1, \ldots, c_{i-1} = d_{i-1} \quad \text{and} \quad c_i \neq d_i.$$

Take a witness $(\bar{a}, \bar{b})$ for (i, c_i, d_i). Then $t(\bar{c}, \bar{a}, \bar{b})$ is a better approximation.

Corollary. *The compact representation of* $\mathbf{P}$ *generates* $\mathbf{P}$ *as a subalgebra.*

Maltsev relational clones

Corollary. $\mathbf{B}^n$ *has at most exponentially many subalgebras (few subpowers).*

Lemma (Dalmau, 2004)**.** *Given the compact representations of* $\mathbf{P}$ *and* $\mathbf{S}$*, then the compact representation of*

- $\mathbf{P} \times \mathbf{S}$*, and*
- $\mathbf{P} \cap \mathbf{S}$

can be computed in polynomial time.

Lemma. *Given the compact representations of* $\mathbf{P}_1, \dots, \mathbf{P}_k$*, and assume that* $P = P_1 \cup \cdots \cup P_k$ *is a subuniverse of* $\mathbf{B}^n$*, then the compact representation of* $\mathbf{P}$ *can be computed in polynomial time.*

Corollary. *Given the compact representation of the relations in* $\mathcal{R}$*, then the compact representation of any relation defined by a primitive positive formula with relations in* $\mathcal{R}$ *can be computed in polynomial time.*

Problem. Can the compact representation of $\mathrm{Sg}(P_1 \cup \cdots \cup P_k)$ be computed in polynomial time?

Few subpowers

Definition. An algebra $\mathbf{B}$ has **few subpowers**, if there exists a polynomial $p(n)$ such that $|S(\mathbf{P}^n)| \leq 2^{p(n)}$ for all n.

- algebras with a Maltsev term $t(y,y,x) \approx t(x,y,y) \approx x$
- algebras with a near-unanimity term $t(y,x,\dots,x) \approx \cdots \approx t(x,\dots,x,y) \approx x$.

Theorem (Idziak, Marković, McKenzie, Valeriote, Willard, 2007)**.** *An algebra* $\mathbf{B}$ *has few subpowers if and only if it has an* **edge** *term* t *satisfying the equations*

$$
\begin{aligned}
t(y,y,x,x,x,\dots,x,x) &\approx x \\
t(x,y,y,x,x,\dots,x,x) &\approx x \\
t(x,x,x,y,x,\dots,x,x) &\approx x \\
&\vdots \\
t(x,x,x,x,x,\dots,x,y) &\approx x.
\end{aligned}
$$

We have compact representations and similar algorithms for few subpower algebras.

Combined algorithm

Theorem. *Let* $\mathbb{B}$ *be a finite relational structure,* $\mathbf{B}$ *be the corresponding algebra on the same universe with all polymorphisms of* $\mathbb{B}$ *as basic operations, and* β *be a congruence of* $\mathbf{B}$*. If* $\mathbf{B}/\beta$ *has few subpowers and the induced algebras on the* β*-blocks generate* $\mathrm{SD}(\wedge)$ *varieties, then* $\mathrm{CSP}(\mathbb{B})$ *is in* $\mathbf{P}$*.*

"Few subpowers above β and bounded width below β."

Definition. $\mathbf{B}$ is an algebra, $\mathbf{B}/\beta$ is Maltsev. The system

$$\mathcal{M} = \{ M_{ij} \leq \mathbf{B}^2 \times (\mathbf{B}/\beta)^n \mid i, j \in A \}$$

is a **Maltsev strategy**, if

- if $(a, b, \bar{c}) \in M_{ij}$ then $a/\beta = c_i$ and $b/\beta = c_j$,
- if $i = j$ then $a = b$,
- $M_{ik} \subseteq \underbrace{\{ (a, b, \bar{c}) \mid \exists d \ (a, d, \bar{c}) \in M_{ij}, (b, d, \bar{c}) \in M_{jk} \}}_{M_{ij} \circ M_{jk}}$.

Consistency algorithm: $M'_{ik} = M_{ik} \cap (M_{ij} \circ M_{jk})$.

Tractable algebras

- bounded width
- few subpowers
- any finite product of bounded width and few subpower algebras
- subuniverses of tractable algebras
- homomorphic images of tractable algebras

Corollary. *Let $\mathcal{V}$ be an idempotent variety. Then every member of the subpseudovariety generated by the bounded width and few subpowers algebras in $\mathcal{V}$ is tractable.*

The "few subpowers below β and bounded width above β" case is still open.

Theorem (Markovic, McKenzie, 2009)**.** *If $\mathbf{B}/\beta$ is a semilattice with*

- *a chain order, or*
- *a flat semilattice order,*

and every β-block is Maltsev, then $\mathbf{B}$ is tractable.

Thank you!